Abstract

This report describes a geometric catalogue of the multi-finger equilibrium grasps which form the basic building block for synthesizing secure robot grasps. Focusing on planar grasps, the report reviews the two and three-finger grasps, then describes graphical techniques for identifying and synthesizing four-finger equilibrium grasps as well as grasps that involve higher number of fingers. For each of these grasps, the report describes the special geometric properties associated with frictionless and frictional finger contacts. These include grasp security properties such as object immobilization and wrench resistance that influence the selection of a hand mechanism design for a given application. The report establishes that when a planar object is held by four fingers, the grasp can always be realized with finger forces that act along the grasped object contact normals. Such grasps are maximally robust with respect to uncertainty in the amount of friction at the contacts. Every planar equilibrium grasp can therefore be realized by two or three finger forces that rely on friction effects, or by four finger forces that need not rely on friction effects. The analogous properties of the 3D equilibrium grasps are also summarized.

1. Introduction

As robot hand mechanisms evolve with innovative designs and grasp planning tools, there is a need for techniques that can identify and synthesize secure multi-finger robot grasps. Equilibrium grasps form the most basic component of such grasps, since any grasped object must be secured by the robot fingers while performing tasks such as object transport, part assembly, and manipulation of hand held tools [1, 2, 3]. This report provides a geometric catalog of the planar multi-finger equilibrium grasps. The report reviews familiar techniques for synthesizing two and three-finger equilibrium grasps, then presents techniques for identifying and synthesizing four-finger equilibrium grasp, with special emphasize on their robustness with respect to the amount of friction at the contacts. The report describes special grasp security properties associated with each of these grasps, as well as possible implications for the design of robot hand mechanisms.

The literature on robot grasping can be divided into three main areas. The first group of papers discusses notions of grasp security. Force closure, or wrench resistance, describes grasps whose fingers can resist any external disturbance wrench (i.e. force and torque) that may act on the grasped object [4, 5]. Form closure, or object immobilization, considers grasps in which the fingers can prevent any motion of the grasped object based on rigid body constraints imposed by the surrounding fingers [6, 7, 8, 9]. Other papers in this group describe algorithms for synthesizing force and form closure grasps e.g., [10, 11, 12, 13, 14]. Another grasp security concern is the need to ensure grasp robustness with respect to small finger placement errors. A pri-
mary example is the notion of contact independent regions, which ensures safety margin with respect to small finger placement errors [4, 13, 15, 16].

A second group of papers seeks to develop simple hand mechanisms capable of grasping a wide variety of objects. Notable examples are the single actuator SDM hand [17] which became the three-finger iRobot-Harvard-Yale hand [18], and Omata’s three-finger force amplification hand [19]. These hands employ underactuated finger mechanisms to achieve stable grasping of a wide variety of objects. Another important progress has been the development of underactuated compliant hands, such as the Pisa/IT softhand [20] and the Harvard-Stanford mobile manipulator hand [21], designed with underactuation and passive mechanical adaptation capabilities. Developers of such novel hand mechanisms as well as users of commercial robot hands such as [22] repetitively evaluate their prototypes on a wide variety of objects and tools [3]. The geometric techniques provided in this report can assist hand developers during such hand design evaluations.

A third group of papers strives to develop grasp planners that have provable performance over a wide range of everyday objects. Examples are Graspit [23], OpenGrasp [24], and Simox [25]. In order to select suitable finger contacts along an object surface, these grasp planners typically optimize various grasp quality measures [26]. For instance, a commonly used grasp quality measure is the maximum volume wrench ellipsoid that can be generated by finite magnitude finger forces [27, 28, 29]. Equilibrium grasp feasibility forms the basic constraint during grasp optimization. The equilibrium grasp techniques described in this report can therefore help develop more efficient grasp planners.

This report describes graphical techniques to assess equilibrium feasibility of planar grasps involving any number of fingers. The cases of frictionless and frictional contacts are discussed, and special properties that affect grasp security and robustness are noted for each of these grasps. While properties of the two and three-finger grasps are known in the robotics literature [4, 30], the report sheds new light on the four-finger equilibrium grasps. Any four-finger equilibrium grasp whose fingers are essential for maintaining the grasp can be realized by finger forces that act along the grasped object’s contact normals. In other words, every frictional equilibrium grasp of a planar object can either be realized by two or three finger forces that rely on friction effects, or by four finger forces that need not rely on friction effects. Note that such grasps are maximally robust with respect to uncertainty in the amount of friction at the contacts. This insight suggests that four-finger hands offer special advantages over lower number of fingers hands.

The report is structured as follows. Section 2 describes the line geometry characterization of multi-finger equilibrium grasps. Section 3 considers the number of fingers associated with the basic planar equilibrium grasps. Section 4 summarizes well known properties of the two and three-finger equilibrium grasps. Section 5 provides a detailed geometric description of the four-finger equilibrium grasps. Section 6 considers higher number of fingers equilibrium grasps. The concluding section sketches the corresponding properties of the 3D equilibrium grasps. The report also contains two appendices that contain proof details and a characterization of the dimension of the set of frictionless equilibrium grasps associated with each number of finger contacts.

2. Line Geometry of Equilibrium Grasps

This section introduces terminology for multi-finger grasps, then describes the line geometry of the equilibrium grasps. The grasped object is a piecewise smooth planar rigid body denoted \( \mathcal{B} \). The object \( \mathcal{B} \) is held by a multi-finger robot hand via contact points \( x_1, \ldots, x_k \), modeled as hard-point contacts [31]. The robot hand interacts with the object through its fingertip points (Figure 1(a)), modeled as rigid finger bodies denoted \( \mathcal{O}_1, \ldots, \mathcal{O}_k \) (Figure 1(b)). The finger contact forces, denoted \( f_1, \ldots, f_k \), are assumed to satisfy the Coulomb friction model, where each finger force lies in the friction cone: \( C_i = \{ f_i \in \mathbb{R}^2 : f_i \cdot n_i \geq 0 \text{ and } |f_i| \leq \mu_i |f_i \cdot n_i| \} \), where \( \mu_i \) is the coefficient of friction at \( x_i \), \( n_i \) is \( \mathcal{B} \)'s inward unit nor-
Definition 1 A rigid object \( B \) is held via \( k \geq 2 \) contacts in a feasible equilibrium grasp (in the absence of external influences such as gravity) when there exist finger forces in the generalized friction cones, \( f_i \in C_i \) for \( i = 1 \ldots k \), satisfying the condition:

\[
\lambda_1 \left( \hat{f}_1 \times \hat{x}_1 \right) + \cdots + \lambda_k \left( \hat{f}_k \times \hat{x}_k \right) = 0 \quad \lambda_1, \ldots, \lambda_k \geq 0
\]

such that the coefficients \( \lambda_1, \ldots, \lambda_k \) are not all zero.

The formulation (1) can be interpreted as a linear mapping of the finger forces \( f_1, \ldots, f_k \) to the net wrench affecting \( B \), denoted \( w = (f, \tau) \in \mathbb{R}^3 \). The matrix representing this linear mapping can be expressed in terms of the finger force components, \((f_1^t, f_k^t) = (f_i \cdot t_i, f_i \cdot n_i)\), as follows.

Definition 2 The grasp matrix \( G \) represents the net wrench affected by the fingers on \( B \) via the relation \( w = Gf \), where \( f \) is the vector of finger force components. In the case of frictionless contacts, \( f = (f_1^t, \ldots, f_k^t) \), and \( G \) is the \( 3 \times k \) matrix:

\[
G = \begin{bmatrix}
    n_1 & \cdots & n_k \\
    x_1 \times n_1 & \cdots & x_k \times n_k
\end{bmatrix}.
\]

In the case of frictional contacts, \( f = (f_1^t, f_1^t, \ldots, f_k^t, f_k^n) \), and \( G \) is the \( 3 \times 2k \) matrix:

\[
G = \begin{bmatrix}
    t_1 & n_1 & \cdots & t_k & n_k \\
    x_1 \times t_1 & x_1 \times n_1 & \cdots & x_k \times t_k & x_k \times n_k
\end{bmatrix}.
\]

When an object is held at a feasible equilibrium grasp, the null space of \( G \) contains non-zero combinations of feasible finger forces satisfying the relation \( Gf = 0 \). These force combinations form the internal forces of the grasp.

The graphical characterization of the equilibrium grasps will be based on line geometry. The following interpretation of the equilibrium grasps is based on Ponce and Merlet [32]. However, Theorem 1 and Corollary 0.0.1 are original contributions of this paper. In line geometry, the Plücker coordinates of a directed spatial line, \( l \), are given by the pair \((l, p \times l) \in \mathbb{R}^6\), where \( l \) is the unit direction of \( l \) and \( p \) is a point on \( l \) [33, 34]. Similarly, the Plücker coordinates of a directed planar line are given by the pair \((l, p \times l) \in \mathbb{R}^3\), where \( l \) is the unit direction of \( l \), \( p \) is a point on \( l \), and \( p \times l = p^T J l \) such that \( J = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0
\end{bmatrix} \). A finger force \( f_i \) acting on \( B \) at \( x_i \) can be associated with the following force line.

Definition 3 The finger force line of a finger force \( f_i \) acting on \( B \) at \( x_i \) is the line passing through \( x_i \) along the direction \( \hat{f}_i \), specified as \((\hat{f}_i, x_i \times \hat{f}_i)\) in Plücker coordinates.

Figure 1: (a) A rigid object held by four finger mechanisms. (b) The fingertips \( O_1, \ldots, O_4 \) represent the full fingers at the given grasp.
Importantly, the Plücker coordinates of a finger force line can be interpreted as the wrench generated by a unit force \( f_1 \) acting on \( B \) at \( x_i, w_i = (f_1, x_i \times f_1) \). It follows that the equilibrium grasp condition, Eq. (1), is equivalent to linear dependency of the finger force lines in Plücker coordinates, with the additional requirement that the coefficients \( \lambda_1, \ldots, \lambda_k \) must be non-negative.

**Linear subspaces of finger force lines:** The Plücker coordinates of all planar lines span a two-dimensional manifold in \( \mathbb{R}^3 \), since \( l \) must be a unit vector. On the other hand, the finger force lines satisfying the equilibrium grasp condition (1) form linearly dependent vectors in the ambient space \( \mathbb{R}^3 \). These vectors span a linear subspace, whose intersection with the two-dimensional manifold of Plücker coordinates gives a collection of linearly dependent finger force lines. We will use the term linear subspace of lines for such collections of linearly dependent finger force lines.

The line geometric interpretation of the equilibrium grasps is summarized in the next theorem, which is stated for 2D as well as 3D grasps. A set of vectors, \( v_1, \ldots, v_k \) in \( \mathbb{R}^n \) is said to positively span the origin when there exist non-negative scalars \( \lambda_1, \ldots, \lambda_k \) which are not all zero, such that \( \lambda_1 v_1 + \cdots + \lambda_k v_k = 0 \).

**Theorem 1** The following two conditions are necessary for equilibrium grasp feasibility:

(i) The finger force directions must positively span the origin of \( \mathbb{R}^n \), where \( n = 2 \) in 2D grasps and \( n = 3 \) in 3D grasps.

(ii) The finger force lines must be linearly dependent in their Plücker coordinates.

Condition (i) corresponds to the force components of the equilibrium grasp equation (1), while condition (ii) is the line geometric interpretation of equation (1). The two conditions are thus necessary for equilibrium grasp feasibility. Before discussing the catalog of equilibrium grasps, let us establish that the conditions of Theorem 1 are necessary and sufficient for equilibrium grasp feasibility when the object is held by a small number of fingers. The proof of the following corollary appears in Appendix A.

**Corollary 0.0.1** The conditions of Theorem 1 are necessary and sufficient for equilibrium grasp feasibility of generic 2D grasps with \( 2 \leq k \leq 3 \) finger contacts, and generic 3D grasps with \( 2 \leq k \leq 6 \) finger contacts.

The following example shows that the necessary conditions of Theorem 1 are no longer sufficient for equilibrium grasps involving higher number of finger contacts.

**Example:** Figure 2 depicts a polygonal object \( B \) held by four disc fingers, \( O_1, \ldots, O_4 \), in a horizontal plane without gravity. The finger forces act along \( B \)'s inward contact normals, and they positively span the origin of \( \mathbb{R}^2 \) as required by condition (i) of Theorem 1. Four force lines in \( \mathbb{R}^2 \) are always linearly dependent in their Plücker coordinates. The finger force lines thus satisfy condition (ii) of Theorem 1. However, the finger forces cannot generate a zero net torque on \( B \) for any combination of the finger force magnitudes. It follows that conditions (i) and (ii) of Theorem 1 are necessary but not always sufficient for equilibrium grasp feasibility.

3. The Basic Equilibrium Grasps

The catalog of planar equilibrium grasps will focus on the two, three, and four-finger grasps. Each of these cases has an important role in the synthesis of secure grasps as next described. For a grasp to be useful, the fingers must be able to hold the object in a way that can resist external wrench disturbances that may act on the grasped object. This notion of
grasp security, termed wrench resistance, is defined as follows.

**Definition 4** Let a rigid object $B$ be held by finger bodies $O_1, \ldots, O_k$ at a given set of contacts. The grasp is **wrench resistant** (or force closure) if every external wrench that may act on $B$ can be resisted by feasible finger forces that act on $B$ at the contacts.

Wrench resistance can be stated as the requirement that the feasible finger forces span the entire object wrench space, which consists of all net object wrenches $w = (f, \tau) \in \mathbb{R}^3$. The complementary notion of grasp security, **object immobilization**, requires that the grasped object be fully restrained by the finger bodies, as stated in the following definition.

**Definition 5** Let a rigid object $B$ be held by rigid finger bodies $O_1, \ldots, O_k$. The grasp is **immobilizing** (or form closure) if the rigid-body constraints imposed by the finger bodies prevent any local motion of $B$ at the equilibrium grasp.

To achieve wrench resistance, two frictional finger contacts can span the entire object wrench space (and hence resist any external wrench that may act on $B$). To achieve object immobilization, three sufficiently flat finger bodies can prevent any motion of the grasped object, based on a combination of first and second order geometric properties of the contacting bodies. In the case of frictionless contacts, four finger contacts can span the entire object wrench space by varying their force magnitudes along the contact normals. Hence, the planar equilibrium grasps involving two, three, and four finger contacts will be fully described in Sections 4, 5, and 6.

The line geometry characterization of the equilibrium grasps will be based on the following characterization of linear subspaces of planar lines. The following list is adapted from Ponce and Merlet [32].

<table>
<thead>
<tr>
<th>Linear Subspaces of Planar Lines:</th>
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<tbody>
<tr>
<td>1) A single planar line spans a one-dimensional subspace in Plücker coordinates, consisting of the line itself.</td>
</tr>
<tr>
<td>2) Two planar lines span a two-dimensional subspace in Plücker coordinates, consisting of the flat pencil of all planar lines passing through the two lines’ intersection point.</td>
</tr>
<tr>
<td>3) Three planar lines that do not intersect at a common point in $\mathbb{R}^2$ span the full three-dimensional space of Plücker coordinates, representing all planar lines.</td>
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When an object $B$ is held at $k$-contact equilibrium grasp, the finger force lines must be linearly dependent in their Plücker coordinates according to Theorem 1. The finger force lines thus span a $(k-1)$-dimensional linear subspace in Plücker coordinates: a one-dimensional subspace in two-finger equilibrium grasps, a two-dimensional subspace in three-finger equilibrium grasps, and the entire three-dimensional space in four-finger equilibrium grasps. This insight is next applied to the individual grasps involving two, three, and four finger contacts.

**4. The Planar Two-Finger Equilibrium Grasps**

The two-finger equilibrium grasps enjoy a special place in industrial robotics, as these grasps are used by parallel-jaw grippers to pick and place parts during manufacturing operations. Based on line geometry, the force lines at a two-finger equilibrium grasp must lie in a common one-dimensional linear subspace in Plücker coordinates. A one-dimensional subspace corresponds to a single line in $\mathbb{R}^2$. The finger forces must therefore lie on a common line in $\mathbb{R}^2$, which necessarily passes through the two contacts. The following lemma specializes Theorem 1 to the case of two-finger grasps.
Lemma 0.0.2 (2-Finger Grasps) A two-finger grasp of a rigid object \( B \) forms a feasible equilibrium grasp iff there exist feasible finger forces, \((f_1, f_2) \in C_1 \times C_2\), such that the two forces have opposing directions and lie on a common line passing through the two contacts.

Note that the finger force magnitudes are equal at any two-finger equilibrium grasp. Lemma 0.0.2 is next applied to the cases of frictionless and frictional finger contacts.

4.1 Frictionless Two-Finger Equilibrium Grasps

Let us first clarify which finger force directions can be realized at a non-smooth boundary point of \( B \), such as a polygon vertex. The generalized contact normal at such a point is defined as the convex combination of the inward unit normals to the boundary curves meeting at the vertex (Figure 3(a)). At a non-smooth boundary point, any vector from the generalized contact normal can be realized as a physical finger force. The frictionless two-finger equilibria are called antipodal grasps, according to the following definition.

Definition 6 A two-finger grasp of an object \( B \) forms an antipodal grasp when the two contact normals, or two vectors from the generalized contact normals, lie on the line passing through the contacts and have opposing directions.

Every piecewise smooth object possesses two special antipodal grasps, termed the minimal and maximal grasps, as stated in the following proposition.

Proposition 0.0.3 Every piecewise smooth object \( B \) possesses at least two antipodal grasps along its outer boundary, a minimal grasp and a maximal grasp (Figure 3(a)).

The proof of Proposition 0.0.3 appears in Appendix A. The proof studies the extrema of the inter-contact distance function, \( g(x_1, x_2) = \|x_1 - x_2\| \), where \( x_1 \) and \( x_2 \) vary along the object’s boundary. The minimal grasp occurs at the pair \((x_1, x_2)\) which forms a saddle point of \( g \) along the object’s boundary. The maximal grasp occurs at the pair \((x_1, x_2)\) which maximizes the value of \( g \) along the object’s boundary.

Example: Figure 3(a) depicts a triangular object, \( B \), together with the generalized contact normals at its vertices. The minimal grasp occurs at a vertex and an opposing edge of \( B \), where the inter-contact distance function \( g(x_1, x_2) \) has a saddle point. The maximal grasp occurs at two opposing vertices of \( B \), where the function \( g(x_1, x_2) \) has a local maximum. Note that objects such as an ellipse possess exactly two antipodal grasps, which is therefore a tight lower bound on the number of frictionless two-finger equilibrium grasps for any piecewise smooth object in \( \mathbb{R}^2 \).

In general, generic piecewise smooth objects, such as polygonal objects without parallel edges, possess only a finite set of antipodal grasps. This is part of a general dimensionality result discussed in Appendix B and illustrated in the following example.

Example: Figure 3(b) depicts a smooth object, \( B \), having a deep concavity at its center. The object possesses four antipodal grasps: two compressive grasps where the finger forces act inward, and two stretching grasps where the finger forces act outward from within the object’s concavity.
4.2 Frictional Two-Finger Equilibrium Grasps

The frictional two-finger equilibrium grasps occur at pairs of contacts where the friction cones contain opposing forces, such that both forces lie on the line passing through the contacts (Lemma 0.0.2). This condition generically holds along contiguous boundary segments rather than at discrete boundary points. The frictional two-finger equilibria are thus robust with respect to small finger placement errors. The following simple test determines if a candidate two-contact arrangement forms a feasible equilibrium grasp [4].

**Graphical Test for Frictional Two-Finger Equilibrium Grasps:**

Let \( C_i^- \) denote the negative reflection of the friction cone \( C_i \) with respect to \( x_i \) (\( i = 1, 2 \)). If \( x_1 \in C_2 \) and \( x_2 \in C_1 \), or \( x_1 \in C_2^- \) and \( x_2 \in C_1^- \), a two-finger equilibrium grasp is feasible. Otherwise a two-finger equilibrium grasp is not feasible.

**Example:** The condition \( x_1 \in C_2 \) and \( x_2 \in C_1 \) means that the vector pointing from \( x_1 \) to \( x_2 \) lies in \( C_1 \), while the opposite vector lies in \( C_2 \) as depicted in Figure 4(a). The condition \( x_1 \in C_2^- \) and \( x_2 \in C_1^- \) means that the vector pointing from \( x_1 \) to \( x_2 \) lies in \( C_1^- \), while the opposite vector lies in \( C_2^- \) as depicted in Figure 4(b). The grasp of Figure 4(a) is compressive, while the grasp of Figure 4(b) forms a stretching grasp.

A final useful property of the frictional two-finger equilibrium grasps can be obtained from their grasp matrix, \( G \), which maps the finger forces \( f_1, f_2 \) to the net wrench affecting the grasped object \( B \). The grasp matrix is \( 3 \times 4 \) in the case of two frictional contacts, and generically has a full rank of three and a one-dimensional null space. The internal grasp forces in the null space of \( G \), \( N(G) \), are spanned by varying the opposing finger force magnitudes:

\[
N(G) = \text{span} \left( \begin{array}{c} x_2 - x_1 \\ x_1 - x_2 \end{array} \right)
\]

By varying the finger forces in their respective friction cones, the finger forces can resist any external wrench that may act on the grasped object, thus satisfying the wrench resistance security criterion of Definition 4.

5. The Planar Three-Finger Equilibrium Grasps

While two-finger hands form the traditional industrial gripper design, three-finger hands offer significant advantages as all-purpose minimalistic robot hands, designed for grasping planar objects of diverse shape and size (as well as cylindrically shaped 3D objects). Based on line geometry, the planar three-finger equilibrium grasps involve three linearly independent finger force lines, that lie in a common two-dimensional subspace in Plücker coordinates. Since such a subspace forms a flat pencil, the finger force lines must intersect at a common point in \( \mathbb{R}^2 \). Based on this insight, the following lemma characterizes the three-finger equilibrium grasps [15].

**Lemma 0.0.4 (3-Finger Grasps)** A three-finger grasp of a planar object \( B \) forms a feasible equilibrium grasp if there exist feasible contact forces, \( (f_1, f_2, f_3) \in C_1 \times C_2 \times C_3 \), which positively span the origin of \( \mathbb{R}^2 \), such that the finger force lines intersect at a common point in \( \mathbb{R}^2 \).
In the case of a three-finger equilibrium grasp with parallel finger forces,\(^1\) two forces must point in the same direction while the third force must have an opposing direction and lie in the strip bounded by the uni-directional finger force lines.

**Example:** Figure 5(a) depicts a three-finger grasp of a rectangular object along two parallel edges. One finger contacts the bottom edge at \(x_1\), while the other two fingers contact the upper edge at \(x_2\) and \(x_3\). The torque component of the equilibrium grasp equation requires that the line of the finger force \(f_1\) lie in the strip bounded by the uni-directional finger forces \(f_2\) and \(f_3\). Hence, the three-finger grasp depicted in Figure 5(a) forms a feasible equilibrium grasp. Once the opposing force at \(x_1\) moves outside the strip bounded by \(x_2\) and \(x_3\), the fingers cannot generate a zero net torque on \(B\) for any combination of the finger force magnitudes (Figure 5(b)).

5.1 Frictionless Three-Finger Equilibrium Grasps

A simple technique for constructing frictionless three-finger equilibrium grasps is illustrated in Figure 6. Let the inscribed disc be the largest disc contained inside the object \(B\). It can be verified that such a disc generically touches the boundary of \(B\) at two or three isolated points [35, 36]. When the inscribed disc touches the boundary of \(B\) at three points, these points form a three-contact equilibrium grasp as depicted in Figure 6(a). When the inscribed disc touches the boundary of \(B\) at two points, any local splitting of one of the two contacts gives a 3-contact equilibrium grasp.

5.2 Frictional Three-Finger Equilibrium Grasps

In the case of frictional contacts, the conditions of Lemma 0.0.4 attain the following form. At a frictional three-finger equilibrium grasp, the friction cones must contain three forces lines which intersect at a common point, such that their directions positively span the origin of \(\mathbb{R}^2\). The graphical procedure for determining equilibrium feasibility of three-contact grasps will be based on the following lemma (see Appendix A).
Lemma 0.0.5 ([30]) Let a planar object $B$ be held in a frictional three-finger equilibrium grasp. If the finger force directions $\hat{f}_1, \hat{f}_2, \hat{f}_3$ are not all parallel, the contact force magnitudes are given up to a common scaling factor by

$$\lambda_i = \hat{f}_{i+1} \times \hat{f}_{i+2} \quad i = 1, 2, 3$$

where index addition is taken modulo 3.

Graphical Test for Frictional Three-Finger Equilibrium Grasps:

1. Check if the three-contact arrangement contains a feasible two-finger equilibrium grasp. Continue if a two-finger equilibrium is infeasible.
2. Enumerate the non-empty polygons of the intersection $C_1 \cap C_2 \cap C_3$, where $C_i$ is the double friction cone at $x_i$ for $i = 1, 2, 3$.
3. Check if a force-triplet in any of the non-empty polygons of $C_1 \cap C_2 \cap C_3$ positively spans the origin of $\mathbb{R}^2$, using a single-point check in each non-empty polygon.
4. If the test succeeds in any of the polygons, the contact arrangement forms a feasible three-finger equilibrium grasp. Otherwise it is not a feasible equilibrium grasp.

Let us clarify some details of the graphical procedure. Consider the non-empty polygons of the intersection $C_1 \cap C_2 \cap C_3$. The polygons containing force triplets that positively span the origin of $\mathbb{R}^2$ will be termed positive span polygons. When a three-contact arrangement does not contain a feasible two-finger equilibrium grasp, the lines passing through all contact pairs do not cross any non-empty positive span polygon of $C_1 \cap C_2 \cap C_3$. The signs of $\lambda_1, \lambda_2, \lambda_3$ therefore remain invariant in each positive span polygon, and a single-point check of the signs of $\lambda_1, \lambda_2, \lambda_3$ suffices in these polygons. As the non-empty polygons of $C_1 \cap C_2 \cap C_3$ contain all possible concurrency points of the finger force line triplets, the graphical procedure accounts for all possible three-finger equilibrium grasps and is thus a complete procedure.

Example: Consider the three-contact arrangement of the polygonal object $B$ depicted in Figure 7. A two-finger equilibrium is not feasible in this contact arrangement. Hence one proceeds with Step 2 of the procedure. The intersection $C_1 \cap C_2 \cap C_3$ consists of two non-empty polygons, $C_1^− \cap C_2^− \cap C_3$ and $C_1^+ \cap C_2^+ \cap C_3$.
Figure 7: Top view of a feasible 3-contact equilibrium grasp illustrating the graphical procedure for determining equilibrium grasp feasibility.

$C_1^- \cap C_2 \cap C_3$. These polygons share the point $x_2$ as a common vertex. Hence one may apply Step 3 using only this vertex. Since the finger force lines that pass through $x_2$ positively span the origin of $\mathbb{R}^2$, the three-contact arrangement forms a feasible equilibrium grasp.

One last property of the frictional three-finger equilibrium grasps is based on their grasp matrix, $G$, which is $3 \times 6$ in the case of three frictional contacts. Generically $G$ has full rank of three and a three-dimensional null space. The internal grasp forces $f_1, f_2, f_3$ in the null space of $G, N(G)$, are spanned by three pairs of opposing finger forces (which may not lie in the respective friction cones):

$$N(G) = \text{span} \left\{ \left( \begin{array}{c} x_2 - x_1 \\ x_1 - x_2 \\ \vec{0} \end{array} \right), \left( \begin{array}{c} \vec{0} \\ x_3 - x_2 \\ x_2 - x_3 \end{array} \right), \left( \begin{array}{c} x_1 - x_3 \\ \vec{0} \\ x_3 - x_1 \end{array} \right) \right\}$$

The internal grasp forces are able to modulate the finger force magnitudes by a common scaling factor, as well as rotate the finger force directions within their respective friction cones. Much like the case of frictional two-finger equilibrium grasps, the rank of $G$ is equal to $\mathcal{B}$’s wrench space dimension for frictional three-finger grasps. These grasps therefore satisfy the wrench resistance grasp security criterion of Definition 4.

6. The Planar Four-Finger Equilibrium Grasps

The four-finger equilibrium grasps possess two useful properties that make them an attractive robot hand design. First, they can immobilize almost any planar object based on rigid body constraints, without any need to rely on contact friction effects. Second, the immobilizing four-finger grasps are robust with respect to small finger placement errors. These properties make the four-finger hands an important choice when grasping objects under low friction conditions using imprecise grasping systems.

Based on line geometry, the finger force lines should be linearly dependent at a four-finger equilibrium grasp. But the space of Plücker coordinates of all planar lines is equivalent to $\mathbb{R}^3$. Hence, four planar force lines are always linearly dependent in this space. However, this does not mean that every four-contact arrangement forms a feasible equilibrium grasp, since the coefficients $\lambda_1, \ldots, \lambda_4$ represent force magnitudes and should be non-negative at a feasible equilibrium grasp. The following proposition characterizes the four-contact arrangements that form feasible equilibrium grasps (see Appendix A).

**Proposition 0.0.6 (4-Finger Grasps)** A four-finger grasp of a planar object $\mathcal{B}$ forms a feasible equilibrium grasp with all four fingers active iff there exist feasible contact forces, $(f_1, f_2, f_3, f_4) \in C_1 \times C_2 \times C_3 \times C_4$, satisfying the conditions:

(i) The finger forces positively span the origin of $\mathbb{R}^2$.

(ii) Each pair of finger forces generates opposite moments about the intersection point of the other pair of finger force lines.

To apply the proposition, one must verify that opposite moments are generated about all six intersection points of the finger force lines. However, based on the proof of Proposition 0.0.6 in Appendix A, it suffices to check this condition only at two intersection points as follows. First verify that the four finger
forces positively span the origin of $\mathbb{R}^2$. Then split the finger forces into two consecutive pairs according to a counterclockwise ordering of their directions. A four-finger equilibrium is feasible iff each of the two pairs generates opposite moments about the intersection point of the other pair, as illustrated in the following example.

**Example:** Figure 8(a) depicts a polygonal object $\mathcal{B}$ held by four disc fingers in a horizontal environment without gravity. The finger forces act along $\mathcal{B}$'s contact normals and their directions positively span the origin of $\mathbb{R}^2$. Hence one proceeds to split the finger forces into two pairs, according to a counterclockwise ordering of their directions. One pair is $(f_1, f_2)$ while the other pair is $(f_3, f_4)$. Since $f_1$ and $f_2$ generate the same clockwise moment about the intersection point of the lines underlying $f_3$ and $f_4$, the finger forces do not form a feasible equilibrium grasp of $\mathcal{B}$. Figure 8(b) depicts a different four-finger grasp of the same object $\mathcal{B}$. The finger force directions have not changed and still positively span the origin of $\mathbb{R}^2$. The two contact pairs generate opposing net moments about the intersection of the grasped object's boundary while maintaining equilibrium feasibility. The frictionless four-finger equilibrium grasps are thus robust with respect to small finger placement errors. This type of robustness is not shared by the two and three-finger grasps associated with frictionless contacts.

A technique for constructing frictionless four-finger equilibrium grasps is next described in the context of polygonal objects. When a polygon $\mathcal{B}$ is held by two fingers at an antipodal grasp, the fingers are located at two opposing vertices, at a vertex and an opposing edge, or on two parallel edges of $\mathcal{B}$. In all of these cases, the generalized contact normals contain opposite vectors at the antipodal points. These vectors positively span the origin of $\mathbb{R}^2$, and this property is maintained by the normals to the edges of $\mathcal{B}$ meeting at the antipodal points. Hence, any local splitting of the antipodal points into two opposing pairs would yield four contact normals that positively span the origin of $\mathbb{R}^2$. Next consider the line passing through the initial antipodal points, denoted $l$. Split each antipodal point into a pair of points, such that the normals to the edges of each pair intersect on $l$. The two contact pairs generate opposing net wrenches on $\mathcal{B}$, thus resulting in a feasible four-finger equilibrium grasp of $\mathcal{B}$.

**Example:** Consider the two polygonal objects depicted in Figure 9. Two antipodal points are first identified along the boundary of these objects. The maximal grasp in the case of the rectangular object...
(Figure 9(a)), and the minimal grasp in the case of the triangular object (Figure 9(b)). Each contact is next split into a pair of contacts lying on opposite sides of the initial point, such that \( B \)'s inward contact normals intersect on the line \( l \). The resulting contact arrangements form feasible four-finger equilibrium grasps of the two objects. Note that the splitting need not be local in the case of polygonal objects.

An important property of the frictionless four-finger equilibrium grasps is their ability to restrain all motions of \( B \), which is the object immobilization grasp security criterion of Definition 5. Since inter-body penetration is physically infeasible under the ideal rigid body model, an immobilized object is fully secured by the surrounding fingers bodies. In general, object immobilization can only be achieved by contact arrangements that form frictionless equilibrium grasps [8, 9]. While object immobilization with two or three fingers requires suitable contact curvature, object immobilization with four fingers does not depend on the contacting bodies' curvature. Moreover, four-finger immobilization is generic, in the sense that almost every frictionless four-finger equilibrium grasp of a piecewise smooth object fully immobilizes the object.\(^2\) This property is illustrated in the following example.

![Diagram of a frictionless four-finger equilibrium grasp](image)

**Example:** Consider the frictionless four-finger equilibrium grasp of the polygonal object \( B \) depicted in Figure 8(b). It can be verified that the object is completely immobilized by the fingers in this grasp. Next consider the non-feasible equilibrium grasp of the same polygonal object depicted in Figure 10(a). The necessary condition for object immobilization is not satisfied in this grasp, hence the object can locally escape the fingers as illustrated in Figure 10(b).

6.2 Frictional Four-Finger Equilibrium Grasps

Given a candidate four-finger contact arrangement with frictional contacts, we wish to graphically determine if the contact arrangement forms an equilibrium grasp for some choice of feasible finger forces. The graphical technique will be based on the following formula for the finger force magnitudes (see Appendix A).

**Lemma 0.0.7** Let a planar object \( B \) be held at a frictional four-finger equilibrium grasp. If the four finger force lines do not intersect at a common point,\(^3\) up to a common scaling factor their magnitude is given by

\[
\lambda_i = (-1)^i \det \begin{bmatrix}
\hat{f}_{i+1} & \hat{f}_{i+2} & \hat{f}_{i+3} \\
x_{i+1} \times \hat{f}_{i+1} & x_{i+2} \times \hat{f}_{i+2} & x_{i+3} \times \hat{f}_{i+3}
\end{bmatrix}
\]

where index addition is taken modulo 4 for \( i = 1 \ldots 4 \).

To obtain a graphical technique that determines equilibrium feasibility, consider the columns of the \( 3 \times 3 \) matrix in (3). The columns form the Plücker coordinates of the finger force lines associated with the forces \( \hat{f}_{i+1}, \hat{f}_{i+2}, \) and \( \hat{f}_{i+3} \). Hence, \( \lambda_i = 0 \) when the three force lines are linearly dependent in their Plücker coordinates. The geometric characterization of the linear subspaces of finger force lines implies that \( \lambda_i = 0 \) either when the three force lines intersect at a common point in \( \mathbb{R}^2 \), or when two of the three force lines happen to be collinear. Hence, when a four-finger equilibrium grasp does not contain any feasible two or three-finger equilibrium grasp, the signs of \( \lambda_1, \ldots, \lambda_4 \) remain invariant in each connected set of four finger force directions (this notion is clarified below). This sign invariance is the basis for the following graphical procedure for testing equilibrium feasibility of a candidate four-contact arrangement.

\(^2\)When a finger touches a non-smooth boundary point of \( B \), a fingertip with non-zero radius of curvature is required for object immobilization.

\(^3\)In particular, the finger force lines may not be all parallel.
Graphical Test for Frictional Four-Finger Equilibrium Grasps:

1. Check if the contact arrangement contains any two-finger or three-finger equilibrium grasp within the allowed friction cones. Continue if such equilibria are infeasible.
2. Check if the contact normals satisfy the four-finger grasp conditions: (i) the force directions positively span the origin of $\mathbb{R}^2$, and (ii) each pair of finger forces generates opposite moments about the intersection point of the other pair of finger force lines.
3. If the test succeeds, the contact arrangement is a feasible four-finger equilibrium grasp. Otherwise the contact arrangement is not a feasible equilibrium grasp.

To justify the graphical procedure, consider the case where the four friction cones, when placed at a common origin, span non-overlapping directions in $\mathbb{R}^2$. In this case the friction cones can be unambiguously ordered in counterclockwise direction with increasing indices, and the equilibrium feasibility test of Proposition 0.0.6 can be applied by splitting each finger force quadruple into two consecutive pairs.

Let us therefore split the friction cones into two consecutive pairs, $(C_1, C_2)$ and $(C_3, C_4)$, and consider the non-empty polygons of the intersections $C_i \cap C_{i+1}$ and $C_i \cap C_{i+2}$, where $C_i = C_i^+ \cup C_i^-$ is the double friction cone at $x_i$ for $i=1 \ldots 4$. The points of $C_1 \cap C_2$ represent all force directions $(\hat{f}_1, \hat{f}_2) \in C_1 \times C_2$, while the points of $C_3 \cap C_4$ represent all force directions $(\hat{f}_3, \hat{f}_4) \in C_3 \times C_4$. The set product, $(C_1 \cap C_2) \times (C_3 \cap C_4)$, represents all force directions $(\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4) \in C_1 \times C_2 \times C_3 \times C_4$. Beyond Step 1 of the procedure, the signs of $\lambda_1, \ldots, \lambda_4$ remain invariant, and in particular remain positive over all choices of finger force directions within any connected set of $(C_1 \cap C_2) \times (C_3 \cap C_4)$ which contains a feasible equilibrium grasp. Equivalently, the friction cones $C_1$ and $C_2$ generate opposite moments about all points of $C_3 \cap C_4$, while the friction cones $C_3$ and $C_4$ generate opposite moments about all points of $C_1 \cap C_2$ (Figure 11(b)). Hence, any particular choice of four finger force directions from $(C_1 \cap C_2) \times (C_3 \cap C_4)$ would suffice to determine equilibrium feasibility. Since the contact normals always pairwise intersect within $C_1 \cap C_2$ and $C_3 \cap C_4$, one can determine equilibrium feasibility using the four contact normals, as illustrated in the following example.

Example: Consider the elliptical object depicted in Figure 11(a), which is held by four disc fingers via frictional contacts with coefficients of friction $\mu_i = 0.3$ for $i = 1 \ldots 4$. This contact arrangement does not support any two or three-finger equilibrium grasp. Hence, one can verify equilibrium feasibility by checking only the contact normals. Since the contact normals do not positively span the origin of $\mathbb{R}^2$, the contact arrangement is not a feasible equilibrium grasp for any choice of finger force directions within the allowed friction cones. Note that higher friction will eventually allow two-finger and three-finger equilibrium grasps at the given contacts. Next consider a different four-contact grasp of the elliptical object with the same amount of friction, shown in Figure 11(b). This contact arrangement still does not support any two or three-finger equilibrium grasp. The contact normals positively span the origin of $\mathbb{R}^2$ and each pair of contact normals generates opposite moments about the intersection point of the other pair of contact normals. This contact arrangement
The equilibrium grasp equation (1) is solved with \( \hat{\lambda}_i \) at the given contacts, at least one point, \( \hat{f}_i \). Every quadruple of finger force directions forms a feasible four-finger equilibrium grasp. The finger force directions \( \hat{f}_i \) in each friction cone \( C_i \) can be parameterized by the intersection of \( \alpha \) and \( \beta \) forces that act along the object’s contact normals.

Proof: According to Lemma 0.0.7, the finger force magnitudes at a frictional four-finger equilibrium grasp are given up to a common scaling factor \( \lambda \). For \( m \) \( C \)-contact normals at the contacts.

\[
\lambda_i = (-1)^i \det \left[ \begin{array}{ccc} \hat{f}_{i+1} & \hat{f}_{i+2} & \hat{f}_{i+3} \\ x_{i+1} \times \hat{f}_{i+1} & x_{i+2} \times \hat{f}_{i+2} & x_{i+3} \times \hat{f}_{i+3} \end{array} \right]
\]

(4)

It follows that each \( \lambda_i \) is a continuous function of the finger force directions \( \hat{f}_{i+1}, \hat{f}_{i+2}, \hat{f}_{i+3} \). The finger force directions in each friction cone \( C_i \) can be parameterized by the intersection of \( \alpha \) and \( \beta \) forces that act along the object’s contact normals.

Every quadruple of finger force directions forms a point, \( (\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4) \), in the connected set \( C_1 \times C_2 \times C_3 \times C_4 \). Let \( (\hat{f}_1^0, \hat{f}_2^0, \hat{f}_3^0, \hat{f}_4^0) \) correspond to a feasible four-finger equilibrium grasp of \( B \). These force directions can be continuously rotated toward the respective contact normals within the connected set \( C_1 \times C_2 \times C_3 \times C_4 \). This rotation can be thought of as a continuous path, \( \alpha(s) : [0, 1] \to C_1 \times C_2 \times C_3 \times C_4 \), such that \( \alpha(0) = (\hat{f}_1^0, \hat{f}_2^0, \hat{f}_3^0, \hat{f}_4^0) \) and \( \alpha(1) = (n_1, n_2, n_3, n_4) \), where \( n_1, \ldots, n_4 \) are \( B \)'s inward unit normals at the contacts.

If the contact normals do not support a four-finger equilibrium grasp at the given contacts, at least one of the coefficients \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) must be negative when the equilibrium grasp equation (1) is solved with \( \hat{f}_i = n_i \) for \( i = 1 \ldots 4 \). By continuity of each \( \lambda_i \) along \( \alpha(s) \), there must exist \( 0 < s^* < 1 \) such that \( \lambda_j(\alpha(s^*)) = 0 \) for some \( j \in \{1, 2, 3, 4\} \), while \( \lambda_i(\alpha(s^*)) > 0 \) for \( i \neq j \).

Based on (4), the condition \( \lambda_j(\alpha(s^*)) = 0 \) indicates that the three finger force lines with indices \( i \neq j \) intersect at a common point in \( \mathbb{R}^2 \), while \( \lambda_i(\alpha(s^*)) > 0 \) for \( i \neq j \) indicates that their directions positively span the origin of \( \mathbb{R}^2 \). The three finger forces thus form a feasible two-finger or three-finger equilibrium grasp of \( B \) at \( \alpha(s^*) \), contradicting the assumption that no such grasps are attainable at the four contacts. The coefficients \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) therefore retain their positive sign along \( \alpha(s) \), implying that the contact normals form a feasible four-finger equilibrium grasp of \( B \) at the given contacts.

Importantly, the amount of friction at the contacts significantly affects the set of attainable two and three-finger equilibrium grasps along a given object boundary. In particular, many two and three-finger equilibrium grasps can only be achieved with non-normal finger force directions. However, in the case of four-finger grasps, Theorem 2 implies that the presence of friction at the contacts does not affect the set of attainable four-finger equilibrium grasps beyond the set afforded by frictionless contacts. A similar property holds for the 3D equilibrium grasps, as discussed in the concluding section.

7. Higher Numbers of Fingers Equilibrium Grasps

When a planar object is held by more than four fingers, the equilibrium grasp can be maintained with a subset of at most four active fingers. The reducibility of such grasps is based on the following Carathéodory’s theorem [37]. The convex hull of a set of points \( \{p_1, \ldots, p_k\} \) in \( \mathbb{R}^m \) is the convex polytope having these points as vertices.

**Carathéodory’s Theorem:** Let \( \{p_1, \ldots, p_k\} \) be a set of points in \( \mathbb{R}^m \). If a point \( p_0 \in \mathbb{R}^m \) lies in the convex hull of \( \{p_1, \ldots, p_k\} \), it lies in the convex hull of at most \( m + 1 \) points from this set.

In planar grasps, the object wrench space is equivalent to \( \mathbb{R}^3 \) hence \( m = 3 \). When \( B \) is held in equilibrium grasp by \( h > 4 \) fingers, at most four of the finger forces are required to positively span \( B \)'s wrench space origin. One can identify these finger forces based on the following notion of essential fingers. In
order to identify the essential fingers at a given grasp, consider the wrench cone associated with each finger contact.

**Definition 8** The wrench cone associated with a finger contact at \( x_i \) is the set of wrenches that can be generated by feasible finger forces acting on \( B \) at \( x_i \), \( W_i = \{ (f_i, x_i \times f_i) : f_i \in C_i \} \).

Each \( W_i \) forms a cone based at the object’s wrench space origin. The set of all net wrenches that can be affected on \( B \) by \( k \) fingers, termed the net wrench cone, is given by the sum of the individual wrench cones, \( W = W_1 + \cdots + W_k \). When the object \( B \) is held in a feasible equilibrium grasp, the net wrench cone contains a linear subspace which passes through \( B \)’s wrench space origin.\(^4\) A cone based at the origin which does not contain a linear subspace is called a pointed cone. The definition of essential fingers follows.

**Definition 9** Let an object \( B \) be held in equilibrium grasp by finger bodies \( O_1, \ldots, O_k \). A finger \( O_i \) is essential for maintaining the \( k \)-finger equilibrium grasp when the net wrench cone spanned by the remaining \( k - 1 \) fingers forms a pointed cone in \( B \)’s wrench space.

Carathéodory’s theorem is next applied to equilibrium grasps having a high number of fingers.

**Corollary 0.0.8 (\( k > 4 \) Fingers)** If a planar object \( B \) is held in a feasible equilibrium grasp by \( k > 4 \) fingers, the equilibrium grasp can be maintained by at most four active finger forces.

**Proof:** Consider the equilibrium grasp equation (1),

\[
\lambda_1 \left( \hat{f}_1 \times x_1 \right) + \cdots + \lambda_k \left( \hat{f}_k \times x_k \right) = \vec{0} \quad \lambda_i \geq 0
\]  \hspace{1cm} (5)

\(^4\)At an equilibrium grasp \( \sum_{i=1}^{k} \lambda_i \vec{w}_i = \vec{0} \), such that \( \vec{w} = (f_i, x_i \times f_i) \) and \( \lambda_1, \ldots, \lambda_k \geq 0 \). Hence, for any net wrench spanned by \( \vec{w}_1, \ldots, \vec{w}_k \), its negated wrench can also be generated as a positive linear combination of these wrenches, thus forming a linear subspace of wrenches.

Since the finger force magnitudes are not all zero at the equilibrium, \( \sum_{j=1}^{k} \lambda_i > 0 \). Multiplying both sides of (5) by \( 1/\sum_{j=1}^{k} \lambda_j \) gives

\[
\sigma_1 \left( \hat{f}_1 \times x_1 \right) + \cdots + \sigma_k \left( \hat{f}_k \times x_k \right) = \vec{0}
\]

where \( \sigma_i = \lambda_i/\sum_{j=1}^{k} \lambda_j \) for \( i = 1 \ldots k \). Since \( \sigma_1, \ldots, \sigma_k \geq 0 \) and \( \sum_{i=1}^{k} \sigma_i = 1 \), the zero wrench lies in the convex hull of the finger wrenches \( \left( \hat{f}_i, x_i \times \hat{f}_i \right) \) for \( i = 1 \ldots k \). Since the object’s wrench space is equivalent to \( \mathbb{R}^3 \) by Carathéodory’s theorem the zero wrench lies in the convex hull of at most four of these wrenches, implying that the equilibrium grasp can be maintained by at most four active finger forces. \( \Box \)

Any planar \( k \)-finger equilibrium grasp thus requires at most four active finger forces. This property can be used to determine equilibrium feasibility of a candidate \( k \)-finger grasp by checking every quadruple of finger contacts, as illustrated in the following example.

**Example:** Figure 12(a) depicts a polygonal object which is held by six disc fingers via frictionless contacts in a horizontal environment without gravity. One way to see that this contact arrangement forms a feasible equilibrium grasp is to note that the object is immobilized by the disc fingers. Since a frictionless equilibrium grasp is necessary for object immobilization, the contact arrangement must form a feasible equilibrium grasp. Based on Corollary 0.0.8, the grasp can be maintained by at most four active finger forces. The two bottom fingers, \( O_1 \) and \( O_2 \), are essential for maintaining the equilibrium grasp. Each of the remaining four fingers is non-essential and can

![Figure 12: (a) Top view of a six-finger equilibrium grasp which contains non-essential fingers. (b)-(c) Two equilibrium grasps containing four essential fingers.](image-url)
be removed without affecting equilibrium feasibility. Figures 12(b)-(c) show two such grasps obtained by removing non-essential finger pairs.

8. Conclusion

The paper described a catalog of the planar equilibrium grasps involving two, three, four, as well as higher number of fingers. The minimal and maximal grasps were used to establish existence of two-finger equilibrium grasps for any piecewise smooth object. The frictional two-finger equilibrium grasps as well as grasps involving higher number of fingers satisfy the wrench resistant grasp security property. In the case of three fingers, the maximal inscribed disc was used to establish existence of three-finger equilibrium grasps for any piecewise smooth object. Three fingers form an attractive all purpose hand design, since they possess the smallest number of fingers required for object immobilization, without any need to rely on friction effects. A graphical test for frictional three-finger equilibrium grasp feasibility based on the intersection arrangement of the double friction cones at the three contacts was described. If the friction cones at the three contacts do not support any two-finger equilibrium grasp, a single point check in each non-empty polygon of the intersection arrangement suffices to determine three-finger equilibrium grasp feasibility.

The paper next focused on the four-finger equilibrium grasps. A test for equilibrium feasibility of a given set of four finger force lines was described. The paper then used a contact splitting technique to establish existence of four-finger equilibrium grasps for any piecewise smooth object. Moreover, the four-finger equilibrium grasps generically immobilize the grasped object, in a way which is robust with respect to small finger placement errors. A graphical test for frictional four-finger equilibrium grasp feasibility was next described. If the friction cones at the four contacts do not support any two or three-finger equilibrium grasp, it suffices to verify equilibrium feasibility using the four contact normals. This property implies that all planar equilibrium grasps can be realized by two or three active finger forces that possibly rely on friction effects, or by four finger forces that need not rely on friction effects. The four-finger hands thus allow finger forces that act along the contact normals which are maximally robust with respect to uncertainty in the amount of friction at the contacts. Finally, the notion of essential fingers was used to relate higher number of finger equilibrium grasps to the basic two, three, and four finger equilibrium grasps.

An analogous catalog of the 3D equilibrium grasps is currently under preparation. It seems that the basic 3D grasps should involve two, three, four, and seven fingers based on the following considerations. Under a frictional soft finger contact model (where the fingers can apply torques about the contact normals), two-finger grasps provide a limited amount of wrench resistance. Four is the smallest number of fingers required for object immobilization, without any need to rely on friction effects. However, four-finger immobilization requires sufficiently flat finger curvature at the contacts, which might call for soft fingertip designs. The seven-finger equilibrium grasps generically immobilize the grasped object without any requirements on the finger curvatures. Moreover, the seven-finger grasps enjoy the robustness property of the planar four-finger grasps reported in this paper: if all seven fingers are essential for maintaining the equilibrium grasp, the grasp can be realized with the finger forces acting along the respective surface normals, without any need to rely on friction effects. Based on the catalog of the 2D grasps and a forthcoming catalog of the 3D grasps, roboticists are welcome to tackle the challenge of identifying simple robot hand designs that can be mass produced at a reasonable cost, yet provide reliable and secure grasps for the widest possible range of everyday grasping tasks.

Appendix A: Proof Details

This appendix contains proofs of Corollary 0.0.1, Proposition 0.0.3, Lemma 0.0.5, Proposition 0.0.6, and Lemma 0.0.7.

Corollary 0.0.1. The conditions of Theorem 1 are necessary and sufficient for equilibrium grasp fea-
consists of two rows and of \(\bar{\lambda}\). Condition (i) of Theorem 1 implies that the columns and the lower row of \(H\) is linearly dependent on its upper two rows. Since \(\hat{H}\) has \(\lambda\) as a vector \((\lambda_1, \lambda_2, \ldots, \lambda_k)\) such that \(\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0\). Hence \(H\) satisfies the equilibrium equation (1), which indicates equilibrium feasibility of the planar two and three-finger grasps (similar arguments hold for the \(2 \leq k \leq 6\) finger 3D grasps). The next proposition considers the minimal and maximal two-finger equilibrium grasps of a planar object.

**Proposition 0.0.3.** Every piecewise smooth planar object possess at least two antipodal grasps along its outer boundary, a minimal grasp and a maximal grasp.

**Proof sketch:** Let us consider the case where the object \(B\) is a smooth convex body. The object’s outer boundary, \(\text{bdy}(B)\), forms a single closed curve in \(\mathbb{R}^2\). Let the ordered pair \((x_1, x_2)\) denote the position of the finger contacts, such that the finger \(O_1\) touches the object boundary at \(x_1\), while the finger \(O_2\) touches the boundary at \(x_2\). The pair \((x_1, x_2)\) varies in the product set \(\text{bdy}(B) \times \text{bdy}(B)\), which forms a compact two-torus. Let \(g(x_1, x_2) = ||x_1 - x_2||\) be the inter-contact distance. Since \(g(x_1, x_2) = g(x_2, x_1)\), an extremum of \(g\) at \((x_1, x_2)\) has a symmetric extremum at \((x_2, x_1)\). The gradient \(\nabla g\) is well defined in the set \(x_1 \neq x_2\), and the extrema of \(g\) in this set occur at points where \(\nabla g(x_1, x_2) = 0\). It can be verified that for convex objects, the extrema of \(g\) correspond to antipodal grasps of \(B\). Let us therefore focus on showing that \(g\) possesses at least two symmetric pairs of extrema in the set \(x_1 \neq x_2\).

Since \(g\) forms a continuous function on the compact manifold \(\text{bdy}(B) \times \text{bdy}(B)\), it attains a global minimum and maximum on this manifold. The global maximum occurs at the endpoints of \(B\)’s maximum width segment, at \(p_1 = (x_1^0, x_2^0)\) and its symmetric counterpart \(p_2 = (x_2^0, x_1^0)\). This segment determines the object’s maximal grasp.

Let us next invoke the mountain pass theorem [38], which can be stated as follows. Let \(f(x_1, x_2)\) have two isolated local minima at \(p_1, p_2 \in \text{bdy}(B) \times \text{bdy}(B)\). Let \(D_1, D_2 \subset \text{bdy}(B) \times \text{bdy}(B)\) be the basins of attraction of \(p_1\) and \(p_2\), each consisting of the points attracted to the local minimum by the flow of \(-\nabla f\). Let \(D_i\) denote the closure of \(D_i\) for \(i = 1, 2\). If the set \(M = D_1 \cap D_2\) is non-empty, it represents a “mountain range” separating \(D_1\) from \(D_2\), on which \(f\) attains higher values. The mountain pass theorem asserts that if \(M\) is non-empty, \(f\) has a saddle point on \(M\). Moreover, the saddle occurs at the point which minimizes the amount of ascent along all paths connecting \(p_1\) to \(p_2\) while passing through \(M\).

In our case, \(g(x_1, x_2)\) has two “mountain tops” at \(p_1\) and \(p_2\), separated by “valleys.” The saddle of \(g\) occurs at the point which minimizes the amount of descent among all paths connecting \(p_1\) to \(p_2\). The saddle and its symmetric counterpart occur at the endpoints of \(B\)’s minimum width segment, which gives the minimal grasp of \(B\).

The next lemma specifies the finger force magnitudes at a three-finger equilibrium grasp.

**Lemma 0.0.5.** Let a planar object \(B\) be held in a
frictional three-finger equilibrium grasp. If the finger force directions \( \hat{f}_1, \hat{f}_2, \hat{f}_3 \) are not all parallel, the contact force magnitudes are given up to a common scaling factor by

\[
\lambda_i = \hat{f}_{i+1} \times \hat{f}_{i+2} \quad i = 1, 2, 3
\]

where index addition is taken modulo 3.

**Proof:** The three-finger equilibrium grasp condition can be written in matrix form as

\[
\begin{bmatrix}
\hat{f}_1 \\
\hat{f}_2 \\
\hat{f}_3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix} = 0.
\]

Since \( \hat{f}_1, \hat{f}_2, \hat{f}_3 \) are not all collinear, the upper two rows are linearly independent, and the \( 3 \times 3 \) matrix has rank two at the equilibrium grasp. Hence, the solution \((\lambda_1, \lambda_2, \lambda_3)\) spans a one-dimensional null space, which must be orthogonal to the three rows.

The following proposition specifies necessary conditions for equilibrium feasibility of the planar four-finger grasps.

**Proposition 0.0.6.** A four-finger grasp of a planar object \( B \) forms a feasible equilibrium grasp with all four fingers active if and only if there exist feasible contact forces, \((f_1, f_2, f_3, f_4) \in C_1 \times C_2 \times C_3 \times C_4\), satisfying the conditions:

(i) The finger forces positively span the origin of \( \mathbb{R}^2 \).

(ii) Each pair of finger forces generates opposite moments about the intersection point of the other pair of finger force lines.

**Proof:** To prove the necessity of conditions (i) and (ii), assume the four-contact arrangement forms a feasible equilibrium grasp. Since we consider equilibrium grasps with four active finger forces, the contact force magnitudes \( \lambda_1, \ldots, \lambda_4 \) are strictly positive in the equilibrium equation \((1)\). The force components of \((1)\) give condition (i). The torque component of \((1)\), \( \lambda_1(x_1 \times \hat{f}_1) + \cdots + \lambda_4(x_4 \times \hat{f}_4) = 0 \), has two non-zero summands when computed about the intersection point of any two finger force lines. The non-zero summands must have opposite signs, which gives condition (ii).

To prove sufficiency, assume that conditions (i) and (ii) are met. Let the finger force indices be assigned according to a counterclockwise ordering of their directions. When these forces are placed at a common origin, the sector spanned by \((\hat{f}_1, \hat{f}_2)\) is disjoint from the sector spanned by \((\hat{f}_3, \hat{f}_4)\). Let \( l_i \) denote the \( i^{th} \) finger force line. Let \( p_{12} \) denote the intersection point of \( l_1 \) and \( l_2 \), and let \( p_{34} \) denote the intersection point of \( l_3 \) and \( l_4 \). The set of net wrenches generated by varying the magnitudes of each pair of forces, \((\hat{f}_1, \hat{f}_2)\) and \((\hat{f}_3, \hat{f}_4)\), corresponds to a sector of force lines. One sector is based at \( p_{12} \) and bounded by \( l_1 \) and \( l_2 \), the other sector is based at \( p_{34} \) and bounded by \( l_3 \) and \( l_4 \). Since \( f_1 \) and \( f_2 \) generate opposite moments about \( p_{34} \), the sector based at \( p_{12} \) must contain a force line which passes through \( p_{34} \). Similarly, the sector based at \( p_{34} \) must contain a force line which passes through \( p_{12} \). Since the sectors spanned by \((\hat{f}_1, \hat{f}_2)\) and \((\hat{f}_3, \hat{f}_4)\) are disjoint, the two collinear forces must have opposite directions. Since one can freely modulate the magnitude of the opposing forces, there exists a combination of force magnitudes which gives a four-finger equilibrium grasp.

The last lemma specifies the finger force magnitudes at a four-finger equilibrium grasp.

**Lemma 0.0.7.** Let a planar object \( B \) be held at a frictional four-finger equilibrium grasp. If the four finger force lines do not intersect at a common point,\(^5\) up to a common scaling factor their magnitude is given by

\[
\lambda_i = (-1)^i \det \begin{bmatrix}
\hat{f}_{i+1} \\
\hat{f}_{i+2} \\
\hat{f}_{i+3}
\end{bmatrix} \begin{bmatrix}
x_{i+1} \\
x_{i+2} \\
x_{i+3}
\end{bmatrix}
\]

where index addition is taken modulo 4 for \( i = 1 \ldots 4 \).

**Proof:** The equilibrium grasp condition can be written in matrix form as

\[
\begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{bmatrix} = 0
\]

\(^5\)In particular, the finger force lines may not be all parallel.
where \( w_i = (\hat{f}_i, x_i \times \hat{f}_i) \) is the wrench generated by the
finger force \( \hat{f}_i \) acting at \( x_i \) for \( i = 1 \ldots 4 \). Since the
four force lines do not intersect at a common point, some of the four force lines do not intersect
at a common point. Let the three non-intersecting lines be associated with the forces \( f_1, f_2, f_3 \). Since \( w_i = (\hat{f}_i, x_i \times \hat{f}_i) \) are the Plücker coordinates of the
\( i \)’th force line, the non-concurrency of the three force lines implies that \( \det[w_1 \ w_2 \ w_3] \neq 0 \). The system of equations:
\[
\begin{bmatrix}
   w_1 & w_1 & w_3 \\
   w_2 & w_2 & w_3 \\
   w_3 & w_3 & w_3
\end{bmatrix}
\begin{bmatrix}
   \lambda_1 \\
   \lambda_2 \\
   \lambda_3
\end{bmatrix} = -\lambda_4 w_4.
\]
can therefore be solved using Cramer’s rule:
\[
\begin{bmatrix}
   \lambda_1 \\
   \lambda_2 \\
   \lambda_3
\end{bmatrix} = \frac{-\lambda_4}{\det[w_1 \ w_2 \ w_3]} \begin{bmatrix}
   \det[w_2 \ w_3 \ w_4] \\
   \det[w_1 \ w_3 \ w_4] \\
   \det[w_1 \ w_2 \ w_4]
\end{bmatrix}.
\]
Expressing \( \lambda_4 = s \cdot \det[w_1 \ w_2 \ w_3] \) for \( s \in \mathbb{R} \), one obtains formula (6) with \( s \in \mathbb{R} \) as the common scaling factor.

**Proof:** Let us determine the dimension of the set \( \mathcal{E} \)
along smooth boundary segments of \( \mathcal{B} \). Let \( n_i \) denote
\( \mathcal{B} \)’s inward unit normal at the finger contact point
\( x(u_i) \). When \( \mathcal{B} \) is held at a fixed configuration, one
can write the frictionless equilibrium grasp condition as the system of equations in \((u_1, \ldots, u_k)\):
\[
\begin{bmatrix}
   \lambda_1 \left(n(u_1) \cdot x(u_1) \times n(u_1)\right) \\
   \vdots \\
   \lambda_k \left(n(u_k) \cdot x(u_k) \times n(u_k)\right)
\end{bmatrix} = \mathbf{0}
\]
where \( \lambda_1, \ldots, \lambda_k \geq 0 \) and \( x \times n = x^T J_n \) such that
\( J_n = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). Let \( \tilde{U} \)
denote the composite space of parameters \((u_1, \ldots, u_k, \lambda_1, \ldots, \lambda_k) \in \mathbb{R}^{2k} \). The set of equilibrium grasps in \( \tilde{U} \), denoted \( \tilde{E} \), is determined by
a combination of equality and inequality constraints:
\[
\tilde{E} = \left\{ \left( u, \lambda \right) \in \tilde{U} : \sum_{i=1}^{k} \lambda_i \left( n(u_i) \cdot x(u_i) \times n(u_i)\right) = \mathbf{0} \text{ and } \lambda_i \geq 0 \right\}
\]
where \( u=(u_1, \ldots, u_k) \) and \( \lambda=(\lambda_1, \ldots, \lambda_k) \). Let \( \pi: \tilde{U} \to \mathcal{U} \) be the coordinate projection, mapping points
\((u, \lambda) \in \tilde{U} \) to points \( u \in \mathcal{U} \). Importantly, the set \( \mathcal{E} \)
is obtained by projecting \( \tilde{E} \) into \( \mathcal{U} \), \( \mathcal{E} = \pi(\tilde{E}) \).

First consider the case of \( k \geq 3 \) fingers. Eq. (7)
imposes three scalar constraints in \( \mathcal{U} \). Let us verify
that these constraints intersect transversally. The three constraints form a vector-valued mapping, \( f: \tilde{U} \to \mathbb{R}^3 \), which maps the equilibrium grasps in \( \tilde{U} \) to the zero net wrench in \( \mathbb{R}^3 \):
\[
f(u, \lambda) = \sum_{i=1}^{k} \lambda_i \left( n(u_i) \cdot x(u_i) \times n(u_i)\right)
\]
If the three constraints intersect transversally, they define a smooth \((2k - 3)\)-dimensional manifold in \( \tilde{U} \).
Transversal intersection requires that the \( 3 \times 2k \) Jacobi
matrix, \( Df \), have full rank at all points of \( \tilde{E} \).
The Jacobian \( Df \) has the form:
\[
Df = \begin{bmatrix}
   \lambda_1 \frac{\partial}{\partial u_1} n_1 & \cdots & \lambda_k \frac{\partial}{\partial u_1} n_k & n_1 & \cdots & n_k \\
   \lambda_1 \frac{\partial}{\partial u_2} (x_1 \times n_1) & \cdots & \lambda_k \frac{d}{du_2} (x_k \times n_k) & x_1 \times n_1 & \cdots & x_k \times n_k
\end{bmatrix}
\]
Note that \( \frac{\partial}{\partial u_i} n_i \) is a 2-vector while \( \frac{d}{du} x_i \times n_i \) is
a scalar. The last \( k \) columns of \( Df \) are linearly dependent at all points of \( \tilde{E} \), since these columns
represent the finger force lines at the equilibrium

**Appendix B: The Dimension of the Set of Frictionless Equilibrium Grasps**

This appendix characterizes the dimension of the set of frictionless equilibrium grasps of planar objects.
Let the object boundary, \( \partial \mathcal{B} \), be parametrized in
its body frame by a continuous mapping \( x(u):[0, 1] \to \partial \mathcal{B} \). Let \( x(u_i) \) denote the position of the \( i \)’th finger
contact. Contact c-space, denoted \( \mathcal{U} \), is the collection of all \( k \) finger contacts given by \( \mathcal{U} = (u_1, \ldots, u_k) \in \mathbb{R}^k \).
The set of frictionless equilibrium grasps in \( \mathcal{U} \), denoted \( \mathcal{E} \), has the following dimension for the various number of contacts.

**Theorem 3 (Dimension of Frictionless Equilibrium)**
The set of frictionless \( k \)-finger equilibrium grasps of
a planar object \( \mathcal{B} \) is generically a discrete set for two
fingers, a two-dimensional set for three fingers, and
a \( k \)-dimensional set for \( k \geq 4 \) fingers.
grasp. It follows that the rank of $Df$ is gener-
ically $\min\{3, 2k - 1\} = 3$ for $k \geq 3$ finger contacts
(note that $\text{rank}(DF) = 3$ even along polygonal boundary segments). The three constraints thus intersect
transversally in $\bar{U}$ for $k \geq 3$ finger contacts, and their solution set forms a $(2k - 3)$-dimensional manifold
in $\bar{U}$. The set $\bar{E}$ is obtained by intersecting the latter
manifold with the $k$-quadrant $\lambda_1, \ldots, \lambda_k \geq 0$. This intersection may introduce boundary components into
the manifold, but it does not change its dimension. The set $\bar{E}$ thus forms a $(2k-3)$-dimensional manifold
with boundary in $\bar{U}$ for $k \geq 3$ finger contacts.

Consider the projection $\pi : \bar{U} \to U$. The function
$f$ is homogeneous (actually linear) in the parameters
$\lambda_1, \ldots, \lambda_k$. Hence $\bar{E}$ consists of entire rays, each hav-
ing the form $(u_1, \ldots, u_k, s\lambda_1, \ldots, s\lambda_k)$ for $s \geq 0$. The
projection $\pi$ maps each of these rays to a single point
$(u_1, \ldots, u_k)$ in $U$. It follows that the dimension of
$E = \pi(\bar{E})$ is one less than the dimension of $\bar{E}$. This es-
turns that the dimension of $E$ is $2k - 4$ for $k \geq 3$ fin-
ger contacts. However, the dimension of $E$ is bounded
from above by the dimension of $U$, which is $k$ in the
case of planar grasps. The dimension of $E$ is therefore
$\min\{2k - 4, k\}$, which is equal to the dimension of $U$
for $k \geq 4$ finger contacts.

In the case of two-finger grasps, the set $E$ is de-
termined by the scalar equations $n_1 \times (x_1 - x_2) = 0$
and $n_2 \times (x_1 - x_2) = 0$, where $u \times v = u^T J v$. It can be
verified that the two equations intersect transversally
in $U \cong \mathbb{R}^2$, hence $E$ forms a discrete set in the case of
two-finger grasps. $\square$
Bibliography


